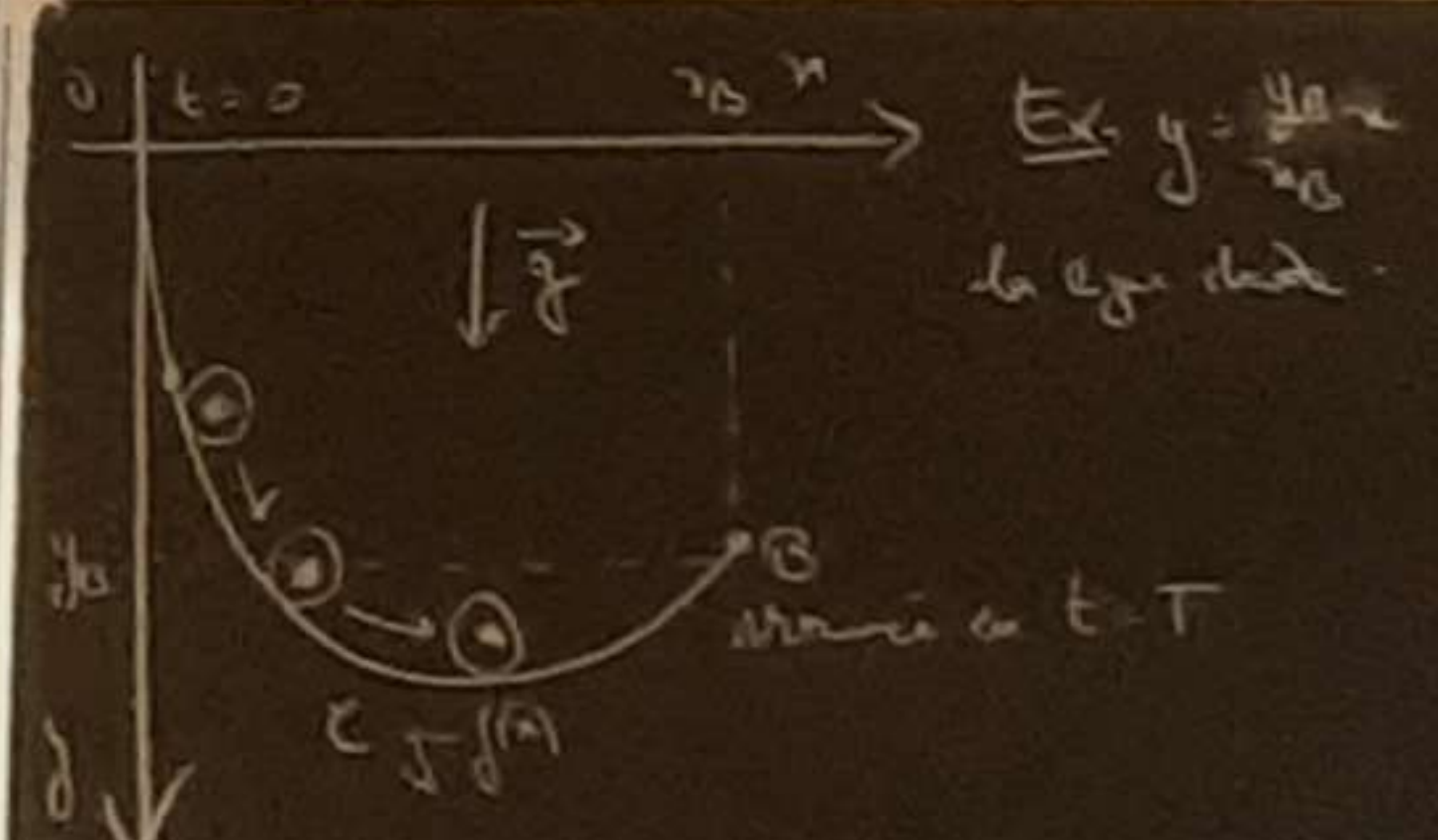


$t=0$ \rightarrow \mathbb{R}^n \rightarrow Ex. $y = \frac{4a-x}{2a}$
 la ligne droite

 $\{f \in C^1([0, 2a]) \cap C^2([0, 2a]) \mid f(0)=0, f(2a)=2a\}$
 $f'(0) > 0$ Δ pas ouvert
 $= E$ (CET)
 Prolongement $T \Leftrightarrow$ Prolongement
 $\mathcal{J} E \rightarrow \mathbb{R} \cup \{\infty\}$
 $f \mapsto \int_0^{2a} \sqrt{\frac{1+f'(x)^2}{f(x)}} dx$
 $L(\mathcal{J}f)$
 car $L: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $(y, y') \mapsto L(y, y') = \sqrt{\frac{1+y'^2}{y}}$
 $\int_0^{2a} \frac{dx}{y}$ car donc $\mathcal{J}(f)$ ligne droite $< \infty$

1) (ADMIS) $f = \arg \min \mathcal{J} \Rightarrow f$ sol de (E \mathcal{L})
 (E \mathcal{L}) : the \mathcal{L} $\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0$
 $0_1 \frac{\partial L}{\partial y} = \frac{L}{2y}$
 $\frac{\partial L}{\partial y'} = \frac{y'}{1+y'^2}$ d'où
 (E \mathcal{L}) : $2ff'' + f'^2 + 1 = 0$
 i.e. $(1+f'(x)^2)f(x) = C$ pour un $C > 0$.
 2) Si L et L' convexes et $\frac{\partial L}{\partial y}$ bornée,
 la réciproque est vraie.
 Soit f_0 sol de (E \mathcal{L}), $g \in E$, $\mathcal{J}(g) < \infty$,
 $f_0 \neq g$.
 $\mathcal{J}(g) - \mathcal{J}(f_0) = \int_0^b (L(g, g') - L(f_0, f_0')) dx$
 $> \int_0^b \langle \nabla L(f_0, f_0') \mid (g - f_0, g' - f_0') \rangle dx$

$\mathcal{J}(g) - \mathcal{J}(f_0) > \int_0^b \left(\frac{\partial L}{\partial y}(f_0, f_0') (g - f_0) + \frac{\partial L}{\partial y'}(f_0, f_0') (g' - f_0') \right) dx$
 $= \int_0^b \left(\frac{\partial L}{\partial y}(f_0, f_0') (g - f_0) + \frac{\partial L}{\partial y'}(f_0, f_0') (g' - f_0') \right) dx$
 $= \left[\frac{\partial L}{\partial y}(f_0, f_0') (g - f_0) \right]_0^b = 0 - 0$
 car $\mathcal{J}(g) > \mathcal{J}(f_0)$ \square
 3) Problème éq. à L convexe.
 $f \in E$ $g = \sqrt{2}f$, $f' = \sqrt{2}g'$
 $\mathcal{J}(f) = \int_0^b \sqrt{\frac{1+g'^2}{g}} dx = \sqrt{2} \int_0^b \sqrt{\frac{1+g'^2}{g}} dx = \sqrt{2} \mathcal{J}(g)$
 car $\mathcal{J} : f \mapsto \int_0^b L(g, g') dx$
 car $\tilde{L} : (y, y') \mapsto \sqrt{\frac{1+y'^2}{y}}$
 et $f = \arg \min \mathcal{J} \Leftrightarrow g = \arg \min \tilde{\mathcal{J}}$

1) \tilde{L} et L convexes (i) $\frac{\partial \tilde{L}}{\partial y}$ bornée
 (ii) $\int_0^b \langle \nabla L(f_0, f_0') \mid (g - f_0, g' - f_0') \rangle dx$
 $g_0 = \sqrt{2}f_0$ vérifie (E \mathcal{L})
 pour \tilde{L} .
 en calculant
 $\frac{\partial \tilde{L}}{\partial y} = -\frac{1}{y^2} \frac{1}{\sqrt{1+y'^2}}$
 $\frac{\partial \tilde{L}}{\partial y'} = \frac{y'}{y \sqrt{1+y'^2}}$ et $\frac{d}{dx} \frac{\partial \tilde{L}}{\partial y'}(g, g')$
 $= \frac{1}{g^2 \sqrt{1+g'^2}} (g g'')$
 Ann :
 (i) $\frac{\partial \tilde{L}}{\partial y} = \frac{y'}{y^2 \sqrt{1+y'^2}} \leq 1$
 (ii) \tilde{L} est strictement convexe.
 En effet,

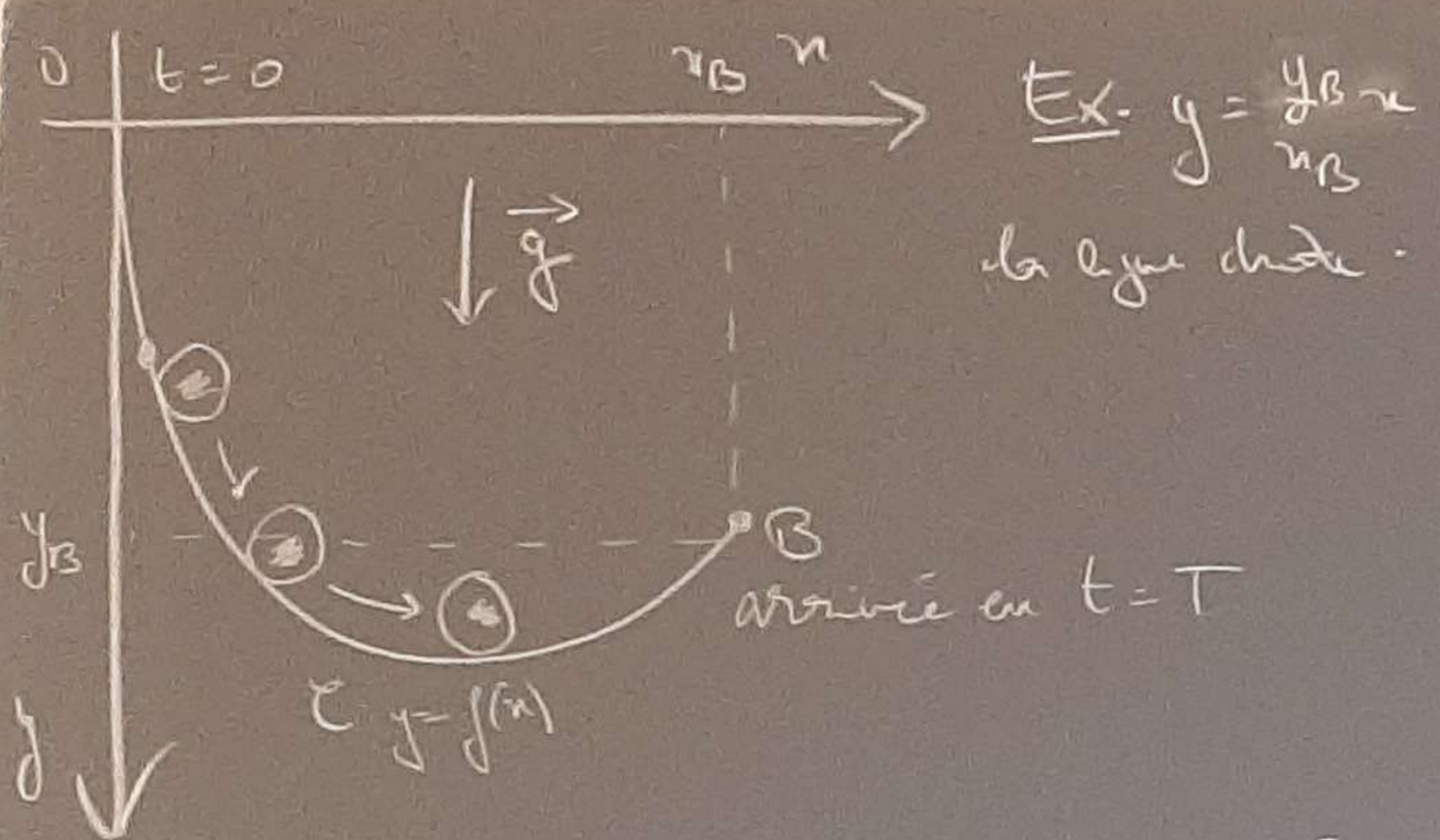
$f \text{ non } (\tilde{L}, y) = \frac{1}{\sqrt{2}} \left(\frac{1}{y^2} \sqrt{2+3y^2} \right)^{1/2}$
 $\in S_2^{++}(\mathbb{R})$.
 car $\begin{cases} \text{Tr}(H(\tilde{L})) > 0 \\ \det(H(\tilde{L})) = \frac{1}{\sqrt{2}} \left(\frac{1}{y^2} \sqrt{2+3y^2} \right) \left(-\frac{y'}{y^2} \right) > 0 \end{cases}$
 (iii)
 $\frac{\partial \tilde{L}}{\partial y}(g_0, g_0') - \frac{d}{dx} \frac{\partial \tilde{L}}{\partial y'}(g_0, g_0')$
 $= -\frac{1}{g_0^2} (g_0'')^2 - \frac{1}{g_0^2} (g_0'')^2 - \frac{1}{g_0^2} (g_0'')^2$
 $= -\frac{1}{g_0^2} (g_0'')^2 \left[\frac{1}{g_0^2} + \frac{1}{g_0^2} + \frac{1}{g_0^2} \right]$
 $g' = \frac{d}{dx} \left(\frac{1}{g} \right) = -\frac{g''}{g^2} + \frac{g'}{g^2} = \frac{1}{g^2} (g' - g'')$

D'où
 $f = \arg \min \mathcal{J} \Leftrightarrow \exists C (1+f'^2)f = C$
 1) Etude qualitative
 Soit f sol du problème.
 (a) f n'est strictement croissante sur aucune intervalle,
 (b) f a au plus 1 point inflexion, lequel est un max.
 (c) f est soit str. \nearrow soit minuscule.
 (d) f est str. \downarrow sur $]\eta, \eta_2]$
 Remarque :
 (a) Par (E \mathcal{L}) et forme de $\frac{\partial L}{\partial y}(f, f')$.

(b) $f(x) = \frac{C}{1+f(x)^2} \leq C$
 et $f(x) = C \Leftrightarrow f'(x) = 0$.
 Soit $f(x_0) = f(x_1) = 0$, et f non str. sur $[x_0, x_1]$
 car f a un maximum au x_2 ,
 $f(x_2) < C$ mais $f'(x_2) = 0$.
 (c) f commence par croître au $f > 0$

x	0	x^*	x_2
$f(x)$	0	+	+
f'	+	0	-
f''	+	-	-

 (d) $f'' = -\frac{1}{2} (1+f^2)^{-1} \frac{1}{f} < 0$



Ex. $y = \frac{y_B}{x_B} x$
la ligne droite.

$\{ f \in C^1([0, x_B]) \cap C^2([0, x_B]) \mid f(0) = 0, f(x_B) = y_B, f(x) > 0 \}$
= E. Δ pas ouvert!
(CEM)

Minimiser $T \Leftrightarrow$ Minimiser $J: E \rightarrow \mathbb{R} \cup \{\infty\}$
 $f \mapsto \int_0^{x_B} \sqrt{\frac{1+f'(x)^2}{f(x)}} dx$

où $L: \mathbb{R}_+^* \times \mathbb{R} \rightarrow \mathbb{R}$
 $(y, y') \mapsto L(y, y') = \sqrt{\frac{1+y'^2}{y}}$

$\mathbb{P}_g \int_0^{x_B} \frac{dx}{\sqrt{x}}$ car donc $J(\text{ligne droite}) < \infty$

① (ADMIS) $f = \text{argmin } J \Rightarrow f$ sol. de (E2)

(E2): the $[0, x_B]$ $\frac{\partial L}{\partial y}(f, f') - \frac{d}{dx} \frac{\partial L}{\partial y'}(f, f') = 0$.

Or $\begin{cases} \frac{\partial L}{\partial y} = -\frac{L}{2y} \\ \frac{\partial L}{\partial y'} = \frac{y' L}{1+y'^2} \end{cases}$ d'où:

(E2): $2y f'' + f'^2 + 1 = 0$
i.e. $(1+f'(x)^2) f(x) = C$ pour un $C > 0$.

② Si L est A-convexe et $\frac{\partial L}{\partial y}$ bornée, la réciproque est vraie.

Soit f_0 sol. de (E2), $g \in E$, $J(g) < \infty$, $f_0 \neq g$.

$J(g) - J(f_0) = \int_0^{x_B} (L(g, g') - L(f_0, f_0')) dx$
 $> \int_0^{x_B} \left\langle \frac{\partial L}{\partial y}(f_0, f_0') \mid (g - f_0) \right\rangle dx$

$J(g) - J(f_0) > \int_0^{x_B} \left[\frac{\partial L}{\partial y}(f_0, f_0')(g - f_0) + \frac{\partial L}{\partial y'}(f_0, f_0')(g' - f_0') \right] dx$

$= \int_0^{x_B} \left(\frac{d}{dx} \frac{\partial L}{\partial y'}(f_0, f_0')(g - f_0) + \frac{\partial L}{\partial y}(f_0, f_0')(g - f_0) \right) dx$
 $= \left[\frac{\partial L}{\partial y'}(f_0, f_0')(g - f_0) \right]_0^{x_B} = 0 - 0$

donc $J(g) > J(f_0)$. \square

③ Problème eq. à L convexe.
 $f \in E$ $g = \sqrt{2} f$, $f' = g g'$

$J(f) = \int_0^{y_B} \sqrt{\frac{1+g^2 g'^2}{\frac{g^2}{2}}} = \sqrt{2} \int_0^{y_B} \sqrt{\frac{1+g'^2}{g^2}} = \sqrt{2} J(g)$

où $\tilde{J}: f \mapsto \int_0^{y_B} \tilde{L}(g, g') dx$

où $\tilde{L}: (y, y') \mapsto \sqrt{\frac{1}{y^2} + y'^2}$

et $f = \text{argmin } J \Leftrightarrow g = \text{argmin } \tilde{J}$

\mathbb{P}_g (i) \tilde{L} str-convexe

(ii) $\delta \int_0^{y_B} \tilde{L}$ pour $g_0 = \sqrt{2} f_0$

on calcule $\begin{cases} \frac{\delta \tilde{L}}{\delta y} = -\frac{1}{y^3} \frac{1}{\tilde{L}} \\ \frac{\delta \tilde{L}}{\delta y'} = y' \frac{1}{\tilde{L}} \end{cases}$

donc:

(ii) $\frac{\delta \tilde{L}}{\delta y'} = \frac{1}{\sqrt{y^2 + y'^2}}$

(i) \tilde{L} str-convexe
En effet.

Ilq (i) \tilde{L} str. convexe (ii) $\frac{\partial \tilde{L}}{\partial y}$ bornée

(iii) Si f_0 vérifie (E2),
par \tilde{L}
 $g_0 = \sqrt{2f_0}$ vérifie (E2)
par \tilde{L} .

on calcule:

$$\begin{cases} \frac{\partial \tilde{L}}{\partial y} = -\frac{1}{y^3} \frac{1}{\tilde{L}} \\ \frac{\partial \tilde{L}}{\partial y'} = y' \frac{1}{\tilde{L}} \text{ et } \frac{d}{dn} \frac{\partial \tilde{L}}{\partial y'}(g, g') \\ = \frac{1}{g^3 \tilde{L}^3} (gg')'' \end{cases}$$

Autre:

(i) $\frac{\partial \tilde{L}}{\partial y'} = \frac{y'}{\sqrt{y'^2 + \frac{1}{y^2}}} \leq 1$

(ii) \tilde{L} str. strictement convexe.

En effet

$$H_{\text{Hess}}(\tilde{L}(y, y')) = \frac{1}{\tilde{L}^3} \begin{pmatrix} \frac{1}{y^3} (\frac{2}{y^2} + 3y^2) & y'/y^3 \\ y'/y^3 & 1/y^2 \end{pmatrix} \in S_2^{++}(\mathbb{R})$$

Car $\begin{cases} \text{Tr}(H(\tilde{L})) > 0 \\ \det(H(\tilde{L})) = \frac{1}{\tilde{L}^6} \left[\frac{1}{y^6} (\frac{2}{y^2} + 3y^2) - \frac{y'^2}{y^4} \right] > 0 \end{cases}$

(iii)

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial y}(g_0, g_0') - \frac{d}{dn} \frac{\partial \tilde{L}}{\partial y}(g_0, g_0') \\ = -g_0^{-3} (g_0^{-2} + g_0'^2)^{-1/2} \\ - g_0^{-3} (g_0^{-2} + g_0'^2)^{-3/2} (g_0 g_0')'' \\ = -g_0^3 (g_0^{-2} + g_0'^2)^{-1/2} \left[g_0^{-2} + g_0'^2 + (g_0 g_0')'' \right] \\ \left(g_0' = \frac{1}{g_0} \right) = (2f_0)^{-1} + \frac{f_0'^2}{2f_0} + f_0'' \end{aligned}$$

$$= \frac{1}{2f_0} (2f_0 f_0'' + f_0'^2 + 1) = 0 \quad \square$$

D'où

$$f = \arg \min J \Leftrightarrow \exists C (1 + f'^2) f = C$$

1) Etude qualitative

Soit f sol. du problème.

- (a) f n'est str. sur aucun intervalle,
- (b) f a au plus 1 point inflexion, auquel cas un max.
- (c) f est soit str. \uparrow soit unimodale
- (d) f' est str. \downarrow sur $[a, b]$

Remarque -

- (a) Par (E2) et forme de $\frac{\partial \tilde{L}}{\partial y}(f, f')$.

(b) $f(n) = \frac{C}{1 + f'(n)^2} \leq C$

et $f(n) = C \Leftrightarrow f'(n) = 0$.

Si $f'(n_0) = f'(n_1) = 0$, et f non str. sur $[n_0, n_1]$ donc a un maximum en n_2 , $f(n_2) < C$ mais $f'(n_2) = 0$.

(c) f commence par croître d'abord car $f > 0$

x	0	n_2	n_3
$f'(x)$	0	+	-
f		→	

x	0	n_2	n_3
$f(x)$	+	0	-
f	→ $f(n_2) = C$		

(d) $f'' = -\frac{1}{2} (1 + f'^2) \frac{1}{f} < 0$